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The Quantum Group $SUq(2)$ and q-Analog of Angular Momentum Operators in Quantum Mechanics

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We present three operators in quantum mechanics that obey the commutation relations of quantum group $SUq(2)$. These operators are nonlinear combinations of the conventional angular momentum operators and are called the quantum q-analog angular momentum operators. When the quantum deformation parameter $r = \ln q$ vanishes, these quantum q-analog angular momentum operators reduce to the usual angular momentum operators.

Recently interest in quantum groups has increased considerably. Quantum groups play an important role in statistical mechanics (Baxter, 1982), inverse scattering problems (Faddeev, 1984), S-matrix theory (Zamolodchikov and Zamolodchikov, 1979), two-dimensional field theories involving fields with intermediate statistics (Frolich, 1987), and conformal theory (Alvarez-Gaume *et al.*, 1989). Drinfeld (1986, 1988) and Jimbo (1985, 1986*a*,*b*) showed that for each representation of quantum groups there is a solution of the Yang–Baxter equation (Yang, 1968; Baxter, 1972). The representation theory of quantum groups, therefore, is of significance. On the other hand, the direct physical application of quantum groups has not yet been accomplished and is of growing interest. The purpose of the present paper is to study the quantum group $SUq(2)$ in nonrelativistic quantum mechanics.

In this paper, we will study the quantum group $SUq(2)$, the q-deformation of Lie algebra $su(2)$, from the point of view of quantum mechanics. It is well known that the basis generators of the Lie algebra $su(2)$ themselves are the usual angular momentum operators in quantum me-

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chanics, which obey the same Lie bracket commutation relations. By analogy to the Jordon–Schwinger realization of Lie algebra *su*(2) (Biedenharn and Louck, 1981), the quantum group $SUq(2)$ has already been studied extensively by Biedenharn (1989), Macfarlane (1989), Sun and Fu (1989), Kulish and Damashinsky (1990), Chaichian *et al.* (1990), and others. However, operators that obey the Lie commutator relations of $SUq(2)$ have not been realized in quantum mechanics. In this paper we construct three q-generators of $SUq(2)$ in terms of conventional angular momentum operators. We will see that the q-generators of $SUq(2)$ can be directly expressed as nonlinear combinations of the original generators of the Lie group $SU(2)$. These q-generators are called quantum q-analog angular momenta (QAAM). We will show that, like the usual angular momenta L_x, L_y, and L_z, the QAAM J_x, J_y, and J_z are Hermitian operators. When the quantum deformation parameter $r = \ln q$ vanishes, QAAM reduce to the usual angular momentum operators in quantum mechanics. It seems that the formulation in our representation of $SUq(2)$ may help in finding some physical model with the structure of the quantum group at the quantum mechanical level.

Let us consider the quantum group $SUq(2)$, which is generated by the operators J_+, J_-, and J_z obeying the Lie bracket relations

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = \frac{\operatorname{sh} r J_z}{\operatorname{sh} \frac{1}{2} r} \tag{2}$$

where we have introduced

$$r = \ln q \tag{3}$$

in which q is a real number. In the limit $r \to 0$, the quantum group $SUq(2)$ contracts to the Lie algebra *su*(2).

It is well known that in quantum mechanics, the angular momentum operators (Schiff, 1968) obey the commutation relations

$$[L_x, L_y] = iL_z, \qquad [L_y, L_z] = il_x, \qquad [L_z, L_x] = iL_y \tag{4}$$

and

$$\begin{aligned} &[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0 \\ &L^2 = L_x^2 + L_y^2 + L_z^2 \end{aligned} \tag{5}$$

where we have chosen $\hbar = 1$. If one introduces

$$L_\pm = L_x \pm iL_y \tag{6}$$

the commutation relations (4) and (5) can be expressed as

$$[L_z, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_z$$
$$[L^2, L_\pm] = [L^2, L_z] = 0 \tag{7}$$

When $r \to 0$, the Lie bracket relations (1) and (2) reduce to (7), i.e., the Lie bracket relations of $su(2)$.

In the following section, we find the expressions for the operators J_+, J_-, and J_z that are formed by combinations of L_+, L_-, and L_z and satisfy the Lie bracket relations (1) and (2). These operators are called quantum q-analog angular momentum operators (QAAM).

Following the above idea, the QAAM operators can be generally expressed as

$$J_\pm = J_\pm(L_\pm, L_z), \qquad J_z = J_z(L_\pm, L_z)$$

Analyzing (1), we may choose

$$J_z = L_z \tag{8}$$

$$J_+ = L_+ \varphi(L^2, L_z) \tag{9}$$

$$J_- = L_- \psi(L^2, L_z) \tag{10}$$

where φ and ψ are two function operators composed by L_z and L^2. By making use of (7) and substituting (8)–(10) into (1), it is easy to verify that J_+, J_-, and J_z satisfy (1) identically.

To study the Lie bracket relation (2), we substitute (8)–(10) into (2) and obtain

$$L_+ \varphi L_- \psi - L_- \psi L_+ \varphi = \frac{\operatorname{sh} r L_z}{\operatorname{sh} \frac{1}{2} r} \tag{11}$$

From (7), we can derive two useful formulas (Qian and Zeng, 1988)

$$\varphi(L^2, L_z)L_- = L_- \varphi(L^2, L_z - 1)$$
$$\psi(L^2, L_z)L_+ = (L_+ \psi(L^2, L_z + 1) \tag{12}$$

Using (11) and (12), we have

$$(L^2 - L_z^2 + L_z)\varphi(L^2, L_z - 1)\psi(L^2, L_z)$$
$$-(L^2 - L_z^2 - L_z)\psi(L^2, L_z + 1)\varphi(L^2, L_z) = \frac{\operatorname{sh} r L_z}{\operatorname{sh} \frac{1}{2} r} \tag{13}$$

We require that the QAAM operators J_+ and J_-, like the usual angular momentum operators $(L_\pm)^+ = L_\mp$, satisfy the conjugate relations

$$(J_+)^+ = J_-, \qquad (J_-)^+ = J_+ \tag{14}$$

Then from (9), (10), and (12) one can deduce

$$\psi(L^2, L_z) = \varphi(L^2, L_z - 1)$$

and

$$J_+ = L_+ \varphi(L^2, L_z), \qquad J_- = L_- \varphi(L^2, L_z - 1) \tag{15}$$

Substituting (15) into (13), we obtain

$$(L^2 - L_z^2 + L_z)\varphi^2(L^2, L_z - 1) - (L^2 - L_z^2 - L_z)\varphi^2(L^2, L_z) = \frac{\operatorname{sh} r L_z}{\operatorname{sh} \frac{1}{2} r} \tag{16}$$

For simplicity we introduce

$$f(L_z) = (L^2 - L_z^2 - L_z)^{1/2} \varphi(L^2, L_z) \tag{17}$$

Equation (16) can be written as

$$f^2(L_z - 1) - f^2(L_z) = \frac{\operatorname{sh} r L_z}{\operatorname{sh} \frac{1}{2} r} \tag{18}$$

In (17), for the operator $A = (L^2 - Lz^2 - Lz)$, an operator $B = A^{1/2}$ is defined as $BB = A$, and $S^{-1/2}$ is defined as the inverse operator of $A^{-1/2}$, i.e., $A^{-1/2}A^{1/2}$ or $A^{1/2}A^{-1/2}$ is the identity operator. Both $A^{1/2}$ and $A^{-1/2}$ are functions of operators L^2 and L_z.

To solve equation (18) for $f(L_z)$, it is convenient to use the hyperbolic function formula

$$\operatorname{ch} rx - \operatorname{ch} ry = 2 \operatorname{sh} \frac{r}{2}(x + y) \operatorname{sh} \frac{r}{2}(x - y)$$

The function $f(L_z)$ that satisfies (17), (18), and the asymptotic behavior $(J_\pm) \xrightarrow[r \to 0]{} L_\pm$ can be found to be

$$f(L_z) = \frac{1}{\operatorname{sh} \frac{1}{2} r} \left\{ \operatorname{sh}^2 \frac{r}{2} L^2 - \frac{1}{2}\left[\operatorname{ch} r\left(L_z + \frac{1}{2}\right) - \operatorname{ch} \frac{r}{2} \right] \right\}^{1/2} \tag{19}$$

Substituting (19) into (15), we obtain the QAAM operators as follows:

$$J_z = L_z \tag{20}$$

$$J_+ = L_+ (L^2 - L_z^2 + L_z)^{-1/2} \frac{1}{\mathrm{sh}\,\frac{1}{2} r} \left\{ \mathrm{sh}^2 \frac{r}{2} L^2 - \frac{1}{2}\left[\mathrm{ch}\, r\left(L_z - \frac{1}{2}\right) - \mathrm{ch}\,\frac{r}{2}\right]\right\}^{1/2} \tag{21}$$

$$J_- = L_- (L^2 - L_z^2 + L_z)^{-1/2} \frac{1}{\mathrm{sh}\,\frac{1}{2} r} \left\{ \mathrm{sh}^2 \frac{r}{2} L^2 - \frac{1}{2}\left[\mathrm{ch}\, r\left(L_z - \frac{1}{2}\right) - \mathrm{ch}\,\frac{r}{2}\right]\right\}^{1/2} \tag{22}$$

Therefore, the quantum group $SUq(2)$ can be directly realized in terms of the usual angular momentum operators in quantum mechanics.

From (20)–(22), it is easy to prove

$$\lim_{r \to 0} J_+ = L_+ \tag{23}$$

$$\lim_{r \to 0} J_- = L_- \tag{24}$$

$$\lim_{r \to 0} J_z = L_z \tag{25}$$

This means that the QAAM operators expressed by (20)–(22) are exactly reduced to the usual angular momentum operators in the limit $r \to 0$. We can also verify that J_+ and J_- defined by (21) and (22) satisfy the conjugate relation (14) precisely.

As the usual angular momentum operators, we define

$$J_x = \frac{1}{2}(J_+ + J_-) \tag{26}$$

$$J_y = \frac{1}{2i}(J_+ - J_-) \tag{27}$$

J_x, J_y, and J_z are the quantum q-analog angular momenta in rectangular component representation. Using (20)–(22), (26), and (27), we have

$$\begin{aligned} J_x = {} & \frac{L_x + iLy}{2\,\mathrm{sh}\,\frac{1}{2} r} (L^2 - L_z^2 + L_z)^{-1/2} \left\{ \mathrm{sh}^2 \frac{r}{2} L^2 - \frac{1}{2}\left[\mathrm{ch}\, r\left(L_z - \frac{1}{2}\right) - \mathrm{ch}\,\frac{r}{2}\right]\right\}^{1/2} \\ & + \frac{L_x + iLy}{2\,\mathrm{sh}\,\frac{1}{2} r} (L^2 - L_z^2 + L_z)^{-1/2} \left\{ \mathrm{sh}^2 \frac{r}{2} L^2 - \frac{1}{2}\left[\mathrm{ch}\, r\left(L_z + \frac{1}{2}\right) - \mathrm{ch}\,\frac{r}{2}\right]\right\}^{1/2} \end{aligned} \tag{28}$$

$$J_y = \frac{L_x + iLy}{2i \,\mathrm{sh}\,\frac{1}{2} r} (L^2 - L_z^2 - L_z)^{-1/2} \left\{ \mathrm{sh}^2 \frac{r}{2} L^2 - \frac{1}{2} \left[\mathrm{ch}\, r\left(L_z - \frac{1}{2}\right) - \mathrm{ch}\,\frac{r}{2} \right] \right\}^{1/2}$$
$$+ \frac{L_x + iLy}{2 \,\mathrm{sh}\,\frac{1}{2} r} (L^2 - L_z^2 + L_z)^{-1/2} \left\{ \mathrm{sh}^2 \frac{r}{2} L^2 - \frac{1}{2} \left[\mathrm{ch}\, r\left(L_z - \frac{1}{2}\right) - \mathrm{ch}\,\frac{r}{2} \right] \right\}^{1/2} \tag{29}$$

$$J_z = L_z \tag{30}$$

Since J_+ and J_- obey equation (14), we see that J_x, J_y, and J_z given by (28)–(30) are Hermitian operators. This can also be directly proved from (28)–(30) by making use of (12).

We conclude that the quantum group $SUq(2)$ is realized in terms of the usual angular momentum operators in quantum mechanics. Three quantum mechanical operators are proposed, which obey the Lie bracket relations of the quantum group $SUq(2)$. These operators are called quantum q-analog angular momenta (QAAM) and are expressed by the nonlinear combination of the usual angular momentum operators. We showed that the rectangular component representations of the QAAM operators are Hermitian. When the quantum deformation parameter $r = \ln q$ vanishes, they reduce to the usual angular momentum operators that satisfy the Lie bracket relations of Lie algebra $su(2)$.

In this paper our discussion has been limited to $SUq(2)$, but it applies to other quantum groups. For example $SUq(N)$ can be similarly realized in terms of the Lie algebra $su(n)$. We will study this problem in another paper.

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